# HYPERSONIC FLOW PAST A CIRCULAR CONE AT AN ANGLE OF ATTACK

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In [1,2], the problem of hypersonic flow past a cone at an angle of attack was solved by expanding the solution in powers of $\varepsilon = (\gamma - 1)/(\gamma + 1)$ and $\sigma = \sin\alpha/\sin\tau$, where $\gamma$ is the ratio of specific heats, $\tau$ is the half vertex angle of the cone and $\alpha$ is the angle of attack; the second approximation was obtained. However, the solution so obtained possesses a logarithmic singularity on the surface of the cone, which indicates the invalidity of the solution near this surface. The correction of the first order approximation in the "vorticity layer" was given by Cheng [1].

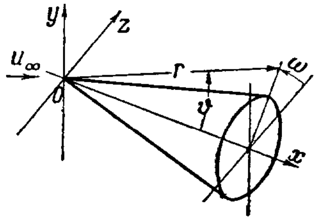
Below we shall present a solution of the problem by successive approximations, which permits a uniform approximation to the exact solution in the entire region between the shock surface and the cone surface, including the "vorticity layer". The solution obtained is accurate up to the second order. It is compared with the results of [1,2].

The inviscid hypersonic flow past conical bodies was studied by Gonor [3,4], who sought solutions as power series in $\varepsilon = (\gamma - 1)/(\gamma + 1)$ and obtained the zeroth order approximation.

However, it is possible to show that the first approximation found by the method of small parameters has a logarithmic singularity on the cone surface. This indicates that this method is inapplicable near the cone surface (see also [5]).

Below, we shall apply the method of Poincaré-Lighthill-Kuo (PLK) to this problem. For a circular cone at an arbitrary angle of attack, we obtain the zeroth order approximation, valid everywhere between the shock surface and the cone surface. It will be shown that outside the vorticity layer, the solution passes over to the solution of Gonor.

1. We consider the uniform flow of a gas past a circular cone of half vertex angle $\tau$ at an arbitrary angle of attack $\alpha$; we choose spherical coordinates $r$, $\vartheta$, and $\omega$ with the axis coinciding with that of the cone (Fig.).




We follow the notation of [1]. We denote by $u_+$, $v_+$ and $w_+$ the components of the velocity vector in the $r$, $\vartheta$, and $\omega$ directions respectively, and by $p_+$ and $\rho_+$ the pressure and density respectively.

We now introduce the dimensionless variables

$$u = \frac{u_+}{u_\infty}, \qquad v = \frac{v_+}{\varepsilon u_\infty \sin\tau}, \qquad w = \frac{w_+}{u_\infty \sin\alpha}$$
$$\rho = \frac{\varepsilon\rho_+}{\rho_\infty}, \qquad p = \frac{p_+}{\rho_\infty u_\infty^2 \sin^2\tau}, \qquad \theta = \frac{\sin\vartheta - \sin\tau}{\varepsilon\sin\tau} \tag{1.1}$$

Then the equations of momentum, continuity, and energy become [1]

$$\left[Iv\frac{\partial}{\partial\theta} + \sigma\frac{w}{1+\varepsilon\theta}\frac{\partial}{\partial\omega}\right]u = \sin^2\tau\,[e^2 v^2 + \sigma^2 w^2]$$
$$\frac{\partial P}{\partial\theta} = \sigma^2\rho\frac{w^2}{1+\varepsilon\theta} - \varepsilon\rho\left[v\frac{\partial}{\partial\theta} + \sigma\frac{w}{I(1+\varepsilon\theta)}\frac{\partial}{\partial\omega} + \frac{u}{I}\right]v$$
$$\sigma\rho\left[Iv\frac{\partial}{\partial\theta} + \sigma\frac{w}{1+\varepsilon\theta}\frac{\partial}{\partial\omega} + u\right]w = -\frac{\varepsilon}{1+\varepsilon\theta}\left[\frac{\partial p}{\partial\omega} + \sigma I\rho v w\right] \tag{1.2}$$
$$2(1+\varepsilon\theta)\rho u + I\frac{\partial}{\partial\theta}[(1+\varepsilon\theta)\rho v] + \sigma\frac{\partial}{\partial\omega}(\rho w) = 0$$
$$\left[Iv\frac{\partial}{\partial\theta} + \sigma\frac{w}{1+\varepsilon\theta}\frac{\partial}{\partial\omega}\right]\left(\frac{p}{\rho^\gamma}\right) = 0$$

Here

$$I = \cos\vartheta = [1 - \sin^2\tau\,(1+\varepsilon\theta)^2]^{1/2}$$

On the cone surface, we have the kinematic boundary condition

$$v = 0 \quad \text{for} \quad \theta = 0$$

On the shock surface $\theta = \theta^+(\omega)$, the conservation of mass, momentum and energy and the continuity of the tangential velocity component have the following form [1]:

$$I\,[\rho v - I\sigma\sin\omega + (1+\varepsilon\theta^+)\cos\alpha] = \sigma\Theta\,[\rho w - \varepsilon\cos\omega]$$
$$[I^2 + \varepsilon^2\Theta^2](p - k\varepsilon) = [I(1+\varepsilon\theta^+)\cos\alpha - I^2\sigma\sin\omega + \sigma\varepsilon\Theta\cos\omega]^2 - \varepsilon\rho\,[Iv - \sigma\Theta w]^2$$
$$[I^2 + \varepsilon^2\Theta](p/\rho - k)(1+\varepsilon) + \varepsilon^2[Iv - \sigma\Theta w]^2 =$$
$$= [I(1+\varepsilon\theta^+)\cos\alpha - I^2\sigma\sin\omega + \sigma\varepsilon\Theta\cos\omega]^2$$

$$I\sigma\,(w - \cos\omega) + \varepsilon\Theta\,[\varepsilon v - I\sigma\sin\omega + (1 + \varepsilon\theta^+)\cos\alpha] = 0$$

$$u - I\cos\alpha = \sin^2\tau\,(1 + \varepsilon\theta^+)\,\sigma\sin\omega \qquad (1.3)$$

Here

$$k = \frac{\gamma + 1}{\gamma(\gamma - 1)M_\infty^2\sin^2\tau} = \text{const}, \qquad \theta_\omega^+ = \frac{d\theta^+(\omega)}{d\omega}, \qquad \Theta = \frac{\theta_\omega^+}{1 + \varepsilon\theta^+}$$

The quantity $k$ is considered bounded. The Bernoulli integral assumes the form

$$(p / \rho - k)\,(1 + \varepsilon)\sin^2\tau + u^2 + \sin^2\tau\,(\varepsilon^2 v^2 + \sigma^2 w^2) = 1 \qquad (1.4)$$

**2.** Cheng [1] sought the solution in the form of a power series in $\varepsilon$ and $\sigma$; for example, the pressure $p$ is written as

$$p = p_{00} + p_{10}\varepsilon + p_{01}\sigma + p_{20}\varepsilon^2 + p_{11}\varepsilon\sigma + p_{02}\sigma^2 + \ldots, \qquad p_{ij} = p_{ij}(\theta, \omega) \qquad (2.1)$$

and the function $\theta^+(\omega)$, defining the position of the shock wave, as

$$\theta^+(\omega) = \theta_{00} + \theta_{10}\,\varepsilon + \theta_{01}\sigma + \theta_{20}\varepsilon^2 + \theta_{11}\varepsilon\sigma + \theta_{02}\sigma^2 + \ldots, \theta_{ij} = \theta_{ij}(\omega)$$

When the solution is given in this form, $\ln\theta$ will appear in the third approximation in the expression for $S = p\rho^{-\gamma}$, so that the third order solution breaks down at the cone surface ($\theta = 0$), indicating that expansion (2.1) is not valid near the cone surface.

Cheng made a correction in the first order approximation; for $S$ it is [2]

$$S = 1 + k + \varepsilon\,[k + 2\,(1 + k)\ln(1 + k)] + 2\sigma\cos\tau\,(1 - \zeta_0^2)/(1 + \zeta_0^2) \qquad (2.2)$$

$$(\zeta_0 = \theta^{\sigma\varepsilon(1+k)\sec\tau}\tan(\tfrac{1}{2}\omega + \tfrac{1}{4}\pi)$$

From (2.2), it follows that the solution depends on $\varepsilon$ and $\sigma$ in such a way that the power series in $\varepsilon$ and $\sigma$ obtained do not converge uniformly in the region between the shock and cone surfaces.

**3.** For the normal and azimuthal velocity components, we introduce the approximate quantities $v^+$ and $w^+$, such that

$$v - v^+ \leqslant O(\theta^{4/3}), \qquad w - w^+ \leqslant O(\theta^{1/3}) \quad \text{as} \ \theta \to 0 \qquad (3.1)$$

We introduce a function $\zeta$, satisfying the equation

$$v^+\cos\tau\,\partial\zeta / \partial\theta + \sigma\,w^+\partial\zeta / \partial\omega = 0 \qquad (3.2)$$

and boundary conditions

$$\zeta = -\sin w \quad \text{for } \theta = {}^1/_2 (1 + k)$$

(The meaning of the boundary condition will become clear later.) The behavior of a line $\zeta$ = const is that of a streamline. It also has a source at the point $\theta = 0$, $\omega = 1/2\pi$, and the cone surface is the line $\zeta = 1$, which we assume to be independent of $\theta$ and $\zeta$; the equation for $S = p\rho^{-\gamma}$ we shall write in the form

$$\frac{\partial S}{\partial \theta} + \sigma \frac{wv^+ \cos\tau - w^+ v\,(1 + \varepsilon\theta)\, I}{(1 + \varepsilon\theta)\, vv^+ I \cos\tau} \frac{\partial S}{\partial \zeta} \frac{\partial \zeta}{\partial \omega} = 0 \tag{3.3}$$

4. Let us consider the system of equations, consisting of the second, third and fourth equations of (1.2), the Bernoulli integral (1.4) and equation (3.3) with boundary conditions (1.3). $S$ may be written as

$$S = S_{00} + \varepsilon S_{01} + \sigma S_{01} + \varepsilon^2 S_{20} + \varepsilon\sigma S_{11} + \sigma^2 S_{02} + \ldots, \qquad S_{ij} = S_{ij}(\theta, \zeta; \varepsilon, \sigma) = 0 \quad (1)$$

We introduce the expansion

$$Z = Z_{00} + Z_{10}\varepsilon + Z_{01}\sigma + Z_{20}\varepsilon^2 + Z_{11}\varepsilon\sigma + Z_{02}\sigma^2 + \ldots, \quad z_{ij} = z_{ij}(\theta, \omega; \varepsilon, \sigma) = O(1) \quad (4.1)$$

in which form we shall assume the solution $p$, $\rho$, $u$, $v$, $w$ and also the function $\theta^+(\omega)$ defining the shock position.

5. Integrating the continuity equation, and considering the boundedness of the desired quantities in the region between the cone surface and the shock surface, we see that $v = O(\theta)$ as $\theta \to 0$. Then this property must also be possessed by $v_{ij}$.

We shall assume that $\partial\zeta/\partial\omega$ exists everywhere and is bounded outside some neighborhood of the point ($\theta = 0$, $\omega = 1/2\pi$). In what follows, the first and second approximations in the examples satisfy these assumptions.

We introduce the function $\zeta_{n+1}$, satisfying the equation

$$\cos\tau \left(\sum_{i+j=0}^{n} v_{ij}^+ \varepsilon^i \sigma^j\right) \frac{\partial \zeta_{n+1}}{\partial \theta} + \sigma \left(\sum_{i+j=0}^{n+1} w_{ij}^+ \varepsilon^i \sigma^j\right) \frac{\partial \zeta_{n+1}}{\partial \omega} = 0 \tag{5.1}$$

and boundary condition

$$\zeta_{n+1} = -\sin\omega \quad \text{for } \theta = {}^1/_2 (1 + k)$$

It will be shown below that $w_{00}^+ = 0$. We estimate the difference $R = \zeta - \zeta_{n+1}$; it satisfies the equation

$$\frac{\partial R}{\partial \theta} + \sigma \frac{w^+}{v^+ \cos \tau} \frac{\partial R}{\partial \omega} =$$

$$= \frac{\sigma}{\cos \tau} \frac{\partial \zeta_{n+1}}{\partial \omega} \left( v^+ \sum_{i+j=1}^{n+1} w_{ij}^+ \sigma^i \varepsilon^j - w^+ \sum_{i+j=0}^{n} v_{ij}^+ \varepsilon^j \sigma^j \right) \left( v^+ \sum_{i+j=0}^{n} v_{ij}^+ \varepsilon^i \sigma^j \right)^{-1}$$

and the boundary condition

$$R = 0 \quad \text{for } \theta = {}^1/_2 (1 + k)$$

Writing the characteristic equation and integrating it, we have

$$R = \int_{{}^1/_2(1+k)}^{\theta} \frac{\sigma O\,[(\varepsilon + \sigma)^{n+2}]}{\theta \cos \tau} \frac{\partial \zeta_{n+1}}{\partial \omega} d\theta$$

The integral is taken along the lines $\zeta$ = const. It will be shown below that $\partial\zeta_1/\partial\omega$ and $\partial\zeta_2/\partial\omega$ have order $\theta^{\varepsilon\sigma}$ as $\theta \to 0$ outside some neighborhood of the singular point; this is also obvious for higher approximations. Then for $R$, we have everywhere the estimate $R = O[(\varepsilon + \sigma)^{n+1}]$ ($R = 0$ for $\theta = 0$).

From this, it follows that to get a solution up to the $(n + 1)$st order quantities, it suffices to know $\zeta_{n+1}$, since $S_{00}$ = const and does not depend on $\zeta$.

**6.** Substituting the expansion of type (4.1) into the system of equations and the boundary conditions indicated in Section 4, and collecting terms with the same powers in $\varepsilon$ and $\sigma$, we obtain equations for the coefficients of the desired quantities.

Since $v = O(\theta)$ as $\theta \to 0$, and the derivative $\partial\zeta/\partial\omega$ and $I$ are bounded, then considering (3.1), we have the following estimate:

$$\partial S \,/\, \partial \theta = O\,(\theta^{q-1}) \qquad (q \geqslant {}^1/_3) \tag{6.1}$$

From (6.1) it follows that the solution to equation (3.3) does not have a logarithmic singularity. The equations for the remaining quantities are such that if no logarithmic singularities occur in the already known approximations, then they will not occur in the solutions of these equations.

**7.** We give the values of the coefficients in the zeroth approximation for the dimensionless quantities

$$p_{00} = 1, \qquad \rho_{00} = (1 + k)^{-1}, \qquad u_{00} = \cos \tau \tag{7.1}$$

$$v_{00} = -2\theta, \qquad w_{00} = (2\theta)^{1/2} (1 + k)^{-1/2} \cos \omega\,, \quad \theta_{00} = {}^1/_2 (1 + k)$$

$$u_{10} = -\tfrac{1}{2}(1+k)\sin\tau\tan\tau, \qquad u_{01} = -\zeta\sin^2\tau$$

$$v_{10} = \theta(1+k)\tan^2\tau - \theta^2(\tan^2\tau - 1) - \tfrac{4}{3}\theta^3(1+k)^{-1} \tag{7.2}$$

$$v_{01} = 2\left(\sin\tau\tan\tau + \frac{2\cos\tau}{1+k}\right)\int_0^\theta \zeta(t,\omega)\,dt + \left(\frac{2\theta}{1+k}\right)^{3/2}\frac{(1+k)\sin\omega}{3\cos\tau} - \frac{4\cos\tau}{1+k}\theta\zeta$$

$$w_{10} = \cos\omega\left\{2(1+k) + \left(\frac{2\theta}{1+k}\right)^{1/2}\left[\frac{k^2}{2(1+k)} - \frac{15}{8}(1+k) - \frac{1+k}{4}\tan^2\tau\right] + \right.$$

$$\left. + \left(\frac{2\theta}{1+k}\right)^{3/2}\left[\frac{1+k}{8}\tan^2\tau - \frac{3}{8}(1+k)\right] - \left(\frac{2\theta}{1+k}\right)^{5/2}\frac{1+k}{24}\right\} - \sec\tau\left(\frac{2\theta}{1+k}\right)^{1/2}\times$$

$$\times\left[\frac{1+k}{2}\cos\tau\cos\omega + \left(\frac{\sin^2\tau}{\cos\tau} + \frac{2\cos\tau}{1+k}\right)\int_0^{1/2(1+k)}\frac{\partial\zeta}{\partial\omega}\,d\theta + \frac{1+k}{6}\frac{\cos\omega}{\cos\tau}\right]$$

$$w_{01} = -\left(\frac{2\theta}{1+k}\right)\frac{\sin 2\omega}{3\cos\tau} + \left(\frac{2\theta}{1+k}\right)^{1/2}\left\{\frac{1}{4}\frac{\sin 2\omega}{\cos\tau}\sin^2\tau - \cos\omega\left(\frac{\sin^2\tau}{\cos\tau} + \right.\right.$$

$$\left.\left. + \frac{2\cos\tau}{1+k}\right)\left[\frac{1}{2}\int_\theta^{1/2(1+k)}\left(\int_0^\varphi \zeta(t,\omega)\,dt - \varphi\zeta(\varphi,\omega)\right)\frac{d\varphi}{\varphi^2} + \frac{1}{(1+k)}\int_0^{1/2(1+k)}\zeta d\theta\right]\right\}$$

$$p_{10} = \frac{5k+1}{4} - \frac{\theta^2}{1+k}, \qquad p_{01} = -2\sin\omega\cos\tau, \quad \rho_{10} = \frac{k^2-\theta^2}{(1+k)^2} + \frac{1}{4}$$

$$\rho_{01} = -2\cos\tau\,[\zeta + (1+k)\sin\omega]\,(1+k)^{-2}, \quad \theta_{10} = \frac{1}{24}(1+k)^2(7+3\tan^2\tau) - \frac{k^2}{2}$$

$$\theta_{01} = \frac{1+k}{2}\cos\tau\sin\omega + \left[\frac{\sin^2\tau}{\cos\tau} + \frac{2\cos\tau}{1+k}\right]\int_0^{1/2(1+k)}\zeta\,d\theta + \frac{1+k}{6}\frac{\sin\omega}{\cos\tau}$$

Here $\zeta(t, \omega)$ and $\zeta(\varphi, \omega)$ indicate that in the formula $\zeta(\theta, \omega)$, $\theta$ has been given in terms of $t$ or $\varphi$. Let us compute the first approximation for $\zeta$, i.e. $\zeta_1$. From (7.1) and (7.2), it follows that

$$v_{00}^{+} = -2\theta, \quad w_{01}^{+} = 0, \quad w_{10}^{+} = 2(1+k)\cos\omega$$

Then for $\zeta_1$ we have

$$-\theta\cos\tau\frac{\partial\zeta_1}{\partial\theta} + \sigma\varepsilon(1+k)\cos\omega\frac{\partial\zeta_1}{\partial\omega} = 0, \quad \zeta_1 = -\sin\omega \quad \text{for } \theta = \tfrac{1}{2}(1+k)$$

The solution for $\zeta_1$ assumes the form

$$\zeta_1 = (1-\eta_1^2)/(1+\eta_1^2), \; \eta_1 = [2\theta/(1+k)]^{\sigma\varepsilon(1+k)\sec\tau}\tan(\tfrac{1}{2}\omega + \tfrac{1}{4}\pi) \tag{7.3}$$

If in (7.2), we substitute the value $\zeta_1$ for $\zeta$, then we have the first approximation up to all first order quantities inclusive, since $R = \zeta - \zeta_1$

is a first order quantity. The integral of $\zeta_1$ may be estimated by the formula

$$\int_0^t \zeta_1(\theta,\omega)\,d\theta = t\zeta_1(t,\omega) + \varepsilon\sigma(1+k)\sec\tau\int_0^t [1-\zeta_1^2(\theta,\omega)]\,d\theta \qquad (7.4)$$

8. Let us determine the pressure in the second approximation

$$p_{20} = \frac{3}{32}(1+k)^2 - \frac{k^2}{4} + \frac{(1+k)^2}{4}\tan^2\tau - \left(\frac{\theta k}{1+k}\right)^2 - \frac{(2\theta)^2}{16} + \frac{5(2\theta)^3}{24(1+k)} + \theta^2\tan^2\tau - \frac{\theta^3\tan^2\tau}{1+k} - \frac{11\theta^4}{6(1+k)^2} \qquad (8.1)$$

$$p_{11} = \frac{4(1+k)}{15}\frac{\sin\omega}{\cos\tau}\left[\left(\frac{2\theta}{1+k}\right)^{5/2} - 1\right] - \frac{4\cos\tau}{(1+k)^2}\int_\theta^{1/2(1+k)} dt\int_0^t \zeta_1(\varphi,\omega)\,d\varphi +$$

$$+ \frac{2\sin^2\tau}{(1+k)\cos\tau}\,\theta\int_0^\theta \zeta_1(t,\omega)\,dt + \frac{2\theta^2}{1+k}\cos\tau\sin\omega + \left(\frac{6\cos\tau}{1+k} + \frac{2\sin^2\tau}{\cos\tau}\right)\times$$

$$\times\int_0^{1/2(1+k)} \zeta_1(\theta,\omega)\,d\theta - \frac{(1+k)\sin\omega}{2\cos\tau}(2\cos^2\tau - 3) + 2\cos\tau\sin\omega$$

$$p_{02} = \cos 2\tau - \cos^2\omega\,[\cos^2\tau + 1/4 - \theta^2(1+k)^{-2}]$$

Here $\zeta$ is replaced by $\zeta_1$, since we limit ourselves to the second approximation. Moreover, expanding in series of $\varepsilon$ and $\sigma$, we easily show that $p_{11}$ and $\theta_{01}$ differ from their corresponding expressions in [2] in the second order quantities. In what follows, the results of [2] will be used. From equation (3.3), we have

$$S_{20} = \text{const}, \qquad S_{11} = f_1(\zeta), \qquad S_{02} = 2\int_{1/2(1+k)}^{\theta} \frac{\cos\omega}{\sqrt{2\theta(1+k)}}\frac{d\zeta}{d\omega}\,d\theta + f_2(\zeta)$$

The integral is taken along the line $\zeta$ = const.

In the expressions for $S_{11}$ and $S_{02}$ we replace $\zeta$ by $\zeta_1$; this is not permitted for second order approximation. Then for the density and radial velocity, we have

$$\rho_{20} = \frac{1}{1+k}\left\{\theta^2\tan^2\tau - 2\left(\frac{\theta k}{1+k}\right)^2 - \frac{\theta^2}{2} + \frac{5\theta^3}{3(1+k)} - \frac{\theta^3\tan^2\tau}{1+k} - \frac{11\theta^4}{6(1+k)^2} + \frac{\theta^2(9+5k)}{4(1+k)} + \frac{(1+k)^2}{4}\tan^2\tau + \frac{11+38k-5k^2}{32} - \frac{(1+k)(7+3\tan^2\tau)}{12} - \frac{k^3}{(1+k)^2}\right\}$$

$$\rho_{11} = \frac{\sin\omega}{(1+k)\cos\tau}\left\{\frac{4(1+k)}{15}\left[\left(\frac{2\theta}{1+k}\right)^{5/2} - 1\right] + \frac{\sin^2\tau}{2}\left[1 - \frac{(2\theta)^2}{1+k}\right] + \frac{k}{2}\left[\left(\frac{2\theta}{1+k}\right)^2\cos^2\tau + 1\right]\right\} + [\zeta_1 + (1+k)\sin\omega]\frac{\cos\tau}{(1+k)^2}\times$$

$$\times\left(\frac{7+3k}{2}+\frac{2\theta^2}{1+k}\right)+\frac{\zeta_1}{(1+k)\cos\tau}\left[\frac{4k}{(1+k)^2}\cos^2\tau+\frac{1}{3}\right]+$$
$$+2\cos\tau\sin\omega\left[\frac{1}{4}-\frac{k^2+\theta^2}{(1+k)^2}\right]$$

$$\rho_{02}=\frac{(3-k)\,\zeta_1^2\cos^2\tau}{(1+k)^3}+\frac{4\zeta_1\cos^2\tau}{(1+k)^2}\sin\omega-\frac{\cos^2\omega}{1+k}\left[\cos^2\tau+\frac{1}{4}-\left(\frac{\theta}{1+k}\right)^2\right]+$$
$$+\frac{\cos^2\tau}{1+k}-\frac{k\sin^2\tau}{(1+k)^2}-\frac{2\,(1-\zeta_1^2)}{(1+k)^2}\left[1-\left(\frac{2\theta}{1+k}\right)^{1/2}\right]$$

$$u_{20}=\sin\tau\tan\tau\left[\frac{k^2}{2}-\frac{(1+k)^2}{4}\tan^2\tau-\frac{(1+k)^2}{6}-\theta^2\right]$$

$$u_{11}=\frac{(1+k)\sin^2\tau}{2\cos\tau}\left[\frac{\zeta_1}{\cos\tau}\left(\frac{4k+2}{1+k}\cos^2\tau+\frac{1}{3}-\sin^2\tau\right)+4\cos\tau\sin\omega\right]$$

$$u_{02}=\frac{\sin^2\tau}{2\cos\tau}\left\{\sin^2\tau-\zeta_1^2-\frac{2\theta}{1+k}\cos^2\omega-2\,(1-\zeta_1^2)\left[1-\left(\frac{2\theta}{1+k}\right)^{1/2}\right]\right\}$$

Since the zeroth and first approximations in the dimensionless variables for the normal and azimuthal velocities correspond to the first and second approximations in the physical quantities, the coefficients $v_{20}$, $v_{11}$, $v_{02}$, $w_{20}$, $w_{11}$ and $w_{02}$ will not be determined.

**9.** It is necessary to determine $\zeta_2$, since using $\zeta_1$ in the formulas for $\rho_{01}$ and $u_{01}$ it is not possible to compute the second approximation. From the equations for $w_{20}$, $w_{11}$ and $w_{02}$, it follows that their solutions are

$$w_{20}=\cos\omega\,\cos^{-2}\tau\,[{}^1/_2\,(3k+2)\,(1+k)\sin^2\tau-2k^2\cos^2\tau-$$
$$-{}^1/_{30}\,(1+k)\,(22k+7)]+\theta^{1/3}f_1(\theta,\omega)+C_1(\omega)\,\theta^{1/2}+O(\varepsilon,\sigma) \qquad (9.1)$$

$$w_{11}=2\,(1+k)\,\zeta_1\cos\omega\,\,[2\,(1+k)^{-1}\cos\tau+\sin\tau\tan\tau]+$$
$$+(1+k)\sin 2w(\cos\tau-{}^1/_4\cos^{-1}\tau)+\theta^{1/3}f(\theta,\omega)+\theta^{1/2}C_2(\omega)+O(\varepsilon,\sigma)$$

$$w_{02}=\theta^{1/3}f_3(\theta,\omega)+\theta^{1/2}C_3(\omega)$$

Here $\zeta$ has been replaced by $\zeta_1$, and we have considered the fact that $v_{20}$, $v_{11}$ and $v_{02}$ have order $\theta$ as $\theta\to 0$, as was proved in Section 5. The functions $f_1$, $f_2$ and $f_3$ are bounded and their forms are determined by the coefficients $\rho_{20}$, $\rho_{11}$, $\rho_{02}$, $v_{20}$, $v_{11}$, $v_{02}$ and some others, while $C_1(\omega)$, $C_2(\omega)$ and $C_3(\omega)$ are determined from boundary conditions. From (7.2) and (7.4) follow

$$v_{10}{}^+=\theta\,\,(1+k)\tan^2\tau, \qquad v_{01}{}^+=2\zeta_1\theta\,\sin\tau\tan\tau \qquad (9.2)$$

The first terms in formulas (9.1) represent $w_{20}{}^+$, $w_{11}{}^+$ and $w_{02}{}^+$. The equation for $\zeta_2$ assumes the form

$$\cos\tau\,(v_{00}^{+} + \varepsilon v_{10}^{+} + \sigma v_{01}^{+})\partial\zeta_2 / \partial\theta + \varepsilon\sigma\,(w_{10}^{+} + \varepsilon w_{20}^{+} + \sigma w_{11}^{+})\,\partial\zeta_2 / \partial\omega = 0 \qquad (9.3)$$

and the boundary condition is

$$\zeta_2 = -\sin\omega \quad \text{for } \theta = {}^1/_2\,(1+k) \qquad (9.4)$$

Since $\zeta_1$ differs from $\zeta_2$ by terms small to the first order, we may replace $\zeta_1$ by $\zeta_2$ in the coefficients of the derivatives of $\zeta_2$ in equation (9.3). The accuracy of the equation for $\zeta_2$ thus only improves, since in the more accurate solution, we must use $\zeta_2$ instead of $\zeta_1$. The solution is

$$f(\zeta_2, \eta_2) = 0 \qquad (9.5)$$

Here

$$\eta_2 = \left(\frac{2\theta}{1+k}\right)^{\varepsilon\sigma\,(1+k)\sec\tau\,(1+\varepsilon n_1+\sigma\zeta_2 n_2)} \tan({}^1/_2\omega + {}^1/_4\pi)\,(\cos\omega)^{\sigma n_3}\,[1 + O[(\varepsilon+\sigma)^2]]$$

$$n_1 = \frac{{}^1/_2\,(1+k)\,(5k+4)\sin^2\tau - 2k^2\cos^2\tau - {}^1/_{30}\,(1+k)\,(22k+7)}{52\,(1+k)\cos^2\tau} \qquad (9.6)$$

$$n_2 = 2\,(1+k)^{-1}\cos\tau + 2\sin\tau\tan\tau, \qquad n_3 = \cos\tau - {}^1/_4\cos^{-1}\tau$$

The function $f$ is determined from boundary condition (9.4).

We may give an approximate formula for $\zeta_2$ accurate up to the first order. This accuracy is sufficient to construct second approximations. Expanding in a series in $\eta_2$, for $\theta = 1/2(1+k)$, we have

$$\eta_2\,({}^1/_2\,(1+k), \omega) = \tan({}^1/_2\,\omega + {}^1/_4\,\pi) + \sigma n_3 \tan({}^1/_2\,\omega + {}^1/_4\,\pi)\ \ln\cos\omega + O\,(\varepsilon^2, \varepsilon\sigma, \sigma^2) \qquad (9.7)$$

This expansion has accuracy up to first order outside some neighborhood with a radius of first order about the point $\omega = 1/2\pi$. However, in this neighborhood the boundary condition differs from unity by quantities of the second order. We shall write the solution as $\zeta_2 = \varphi(\eta_2)$. Satisfying the boundary condition up to first order quantities, we have

$$\zeta_2 = \frac{1-\eta_2^2}{1+\eta_2^2} + \sigma n_3 \frac{4\eta_2^2}{(1+\eta_2^2)^2} \ln\frac{2\eta_2}{1+\eta^2} + O\,((\varepsilon+\sigma)^2) \qquad (9.8)$$

where $\eta_2$ is given by (9.6), in which $\zeta_2$ is replaced by $\zeta_1$. Thus, $\zeta_2$ has accuracy up to first order inclusive everywhere.

In this manner, we have completely determined the second approximation in the physical variables.

**10.** Using this method, it is possible to construct higher order approximations; but the number of terms in the equations increases rapidly, rendering the computational process complicated.

If the resulting solution is formally expanded in a series in $\varepsilon$ and $\sigma$, then it passes over to the solution obtained from a formal application of the method of small parameters. However, the series thus obtained diverges on the surface of the cone ($\theta = 0$), and in its neighborhood there is no uniform convergence.

To compare the solutions obtained by these two methods, we consider an example with the following data: $\tau = 45^\circ$, $\alpha = 17^\circ$, $M_\infty = 9.4$ and $\gamma = 1.40$.

The difference between the radial velocities computed by the two different methods in the second approximation constitutes 8 per cent at $\theta = 0.1$, 14 per cent at $\theta = 0.01$, 23 per cent at $\theta = 0.001$, and the difference becomes 80 per cent on the surface of the cone.

Similar solutions may be given for other supersonic flows past cones at angles of attack.

**11.** Below we shall consider the inviscid hypersonic flow of a uniform stream past a circular cone with half vertex angle $\tau$ at an arbitrary angle of attack $\alpha$, using a spherical coordinate system $r$, $\vartheta$, and $\omega$ with the axis along the axis of the cone (Fig.).

As independent variables we shall take $\omega$ and $\psi$, where $\psi(\omega, \vartheta)$ satisfies the equation

$$v_+ \partial\psi / \partial\vartheta + w_+ \operatorname{cosec}\vartheta \, \partial\psi / \partial\omega = 0$$

The equations of momentum, continuity and energy assume the form [3]

$$\frac{w_+}{\sin\vartheta}\frac{\partial u_+}{\partial\omega} - v^2_+ - w^2_+ = 0$$

$$\frac{\omega_+}{\sin\vartheta}\frac{\partial v_+}{\partial\omega} + u_+v_+ - \frac{w_4{}^2{}_+}{\tan\vartheta} = -\frac{1}{\rho_+\vartheta_\psi}\frac{\partial p_+}{\partial\psi}, \quad \frac{\partial}{\partial\omega}\ln(\rho_+ w_+ \vartheta_\psi) + 2\frac{u_+}{w_+}\sin\vartheta = 0 \qquad (11.1)$$

$$\frac{\gamma p_+}{\rho_+(\gamma-1)} + \frac{u_+^2 + v_+^2 + w_+^2}{2} = \frac{\gamma p_\infty}{\rho_\infty(\gamma-1)} + \frac{u_\infty^2}{2}$$

$$\frac{\partial}{\partial\omega}\frac{p_+}{\rho_+{}^\gamma} = 0, \qquad w_+\frac{\partial\vartheta}{\partial\omega} = v_+\sin\vartheta$$

The solution must satisfy the conditions on the shock wave and on the cone surface [3].

Gonor solved system (11.1) using the method of expansion in the small parameter $\varepsilon = (\gamma - 1)/(\gamma + 1)$, setting

$$u_+ = u_0 + \varepsilon u_1, + \ldots, v_+ = \varepsilon v_0 + \varepsilon^2 v_1 + \ldots, w_+ = w_0 + \varepsilon w_1 + \ldots$$
$$p_+ = p_0 + \varepsilon p_1 + \ldots, \rho_+ = \varepsilon^{-1}\rho_0 + \rho_1 + \ldots, \vartheta = \tau + \varepsilon\vartheta_0 + \ldots \qquad (11.2)$$

He found the values of the coefficients with the subscript zero. However, if one attempts to find the coefficients with subscript 1, then $w_1$ has a logarithmic singularity on the conical surface.

**12.** We transform system (11.1) to the form which is convenient for the application of the PLK method.

Since $v_+ \sim O(\varepsilon)$, from the first equation in system (11.1) follows

$$\operatorname{cosec} \vartheta \, \partial u_+ / \partial \omega = w_+ + O(\varepsilon^2) \tag{12.1}$$

(This relationship obtains exactly on the cone surface, since on it $v_+ = 0$, and near it $v_+ \sim O(\vartheta - \tau)$.) We write $p_+$ and $\rho_+$ in the following form:

$$\begin{aligned} p_+ &= p_0 + p_1\varepsilon + O(\varepsilon^2), \qquad p_i = p_i(\omega, \psi; \varepsilon) = O(1) \\ \rho_+ &= \varepsilon^{-1}[\rho_0 + \varepsilon \rho_1 + O(\varepsilon^2)], \qquad \rho_i = \rho_i(\omega, \psi; \varepsilon) = O(1) \end{aligned} \tag{12.2}$$

Then from the Bernoulli integral (the fourth equation in (11.1)) for constant entropy along the streamlines, we have

$$u_+^2 + \left(\operatorname{cosec}\vartheta \frac{\partial u_+}{\partial \omega}\right)^2 + \frac{p_0^+}{\rho_0^+}\left[1 + \varepsilon\left(1 + 2\ln\frac{p_0}{p_0^+} + \frac{p_1^+}{p_0^+} - \frac{\rho_1^+}{\rho_0^+}\right)\right] + O(\varepsilon^2) = u_\infty^2 + \frac{2\gamma p_\infty}{(\gamma - 1)\rho_\infty} \tag{12.3}$$

The plus-sign superscript indicates quantities on the intersection lines of the stream surfaces $\psi$ = const with the shock wave.

We introduce a new variable instead of $\omega$

$$z = \int_{\omega^+}^{\omega} \sin \vartheta \, d\omega$$

Moreover, we introduce the notation

$$f(z, \psi) = X_0(\psi) + \varepsilon X_1(z, \psi), \qquad X_0(\psi) = \left[u_\infty^2 + \frac{2\gamma p_\infty}{\rho_\infty(\gamma - 1)} - \frac{p_0^+}{\rho_0^+}\right]^{1/2} \tag{12.4}$$

$$X_1(z, \psi) = -\frac{p_0^+}{2\rho_0^+ + X_0(\psi)}\left[1 + 2\ln\frac{p_0}{p_0^+} + \frac{p_1^+}{p_0} - \frac{\rho_1^+}{\rho_0^+}\right]$$

Then equation (12.3) becomes

$$u_+^2 + \left(\frac{\partial u}{\partial z}\right)^2 = f^2(z, \psi) \tag{12.5}$$

We seek a solution $u_+$ in the form

$$u_+ = f(z, \psi) \sin y(z, \psi) \tag{12.6}$$

Then the equation for $y(z, \psi)$ has the form

$$fy_z' + f_z' \tan y = f(z, \psi) \tag{12.7}$$

The prime indicates differentiation. For the independent variables, we assume

$$z = q + \varepsilon z_1(q, \psi) + \ldots, \psi = \psi \tag{12.8}$$

We seek the unknown function in the form

$$y = y_0(q, \psi) + \varepsilon y_1(q, \psi) + \ldots \tag{12.9}$$

Substituting (12.8) and (12.9) into (12.7), we have

$$X_0 y_{0q}' + \varepsilon (X_1 y_{0q}' + X_0 y_{1q}') + \varepsilon X_{1q}' \tan y_0 + \ldots = X_0 + \varepsilon (X_0 z_{1q}' + X_1) + \ldots \tag{12.10}$$

Collecting the free terms in (12.10), we get

$$y_{0q}' = 1, \quad \text{i.e. } y_0 = q + \alpha(\psi) \tag{12.11}$$

To satisfy the boundary condition on the shock surface [3], we have

$$\alpha(\psi) = \tan^{-1} \frac{u_0^+}{w_0^+}$$

Collecting terms with $\varepsilon$ and requiring that $y_1$ be nonsingular, we get

$$z_{1q}' = X_{1q}' \tan [q + \alpha(\psi)]$$

Then

$$y_{1q}' = 0, \qquad z_1 = \int_0^q X_{1q}' \tan [q + \alpha(\psi)]\, dq, \qquad y_1 = y_1(\psi) \tag{12.12}$$

From (12.12), (12.4) and the boundedness of $X_{1q}'$ follows the estimate

$$z = q + \varepsilon O \{(\partial p_0 / \partial q) \ln \cos [q + \alpha(\psi)]\}$$

From this, it is obvious that $z - q \leqslant O(\varepsilon \ln \varepsilon)$ for $q + \alpha(\psi) \leqslant 1/2\pi - O(\varepsilon)$, where $O(\varepsilon) > 0$. As $q + \alpha(\psi)$ approaches $1/2\pi$, the quantity $z$ increases rapidly for slight increases in $q$.

Furthermore, from (12.1), (12.4), (12.6), (12.11) and (12.12), we have

$$\begin{gathered} u_0 = X_0(\varphi) \sin [q + \alpha(\psi)] \\ u_1 = X_1(z, \psi) \sin [q + \alpha(\psi)] + X_0(\psi) \cos [q + \alpha(\psi)]\, y_1(\psi) \\ w_0 = X_0(\psi) \cos [q + \alpha(\psi)]\, \partial q / \partial z \end{gathered} \tag{12.13}$$

$$w_1 = X_{1z}'(z, \psi) \sin [q + \alpha(\psi)] + X_1 \cos [q + \alpha(\psi)] \partial q / \partial z -$$
$$- X_0 \sin [q + \alpha(\psi)] y_1(\psi) \partial q / \partial z$$

From this, it is clear that $u_0$ and $w_0$ differ in terms of the first order from the corresponding results of Gonor [3] when

$$q + \alpha(\psi) \leqslant {}^1/_2 \pi - O(\varepsilon)$$

i.e. outside the vortex layer.

Carrying out the limiting process on the cone surface, we have

$$u_0 = X_0(\psi_0), \quad u_1 = + X_1(z, \psi_0), \quad w_0 = 0, \quad w_1 = X_1'(z, \psi_0) \tag{12.14}$$

where $\psi = \psi_0$ for the cone surface.

These quantities agree with the results on the cone surface in [5].

**13.** Instead of $\vartheta$ we introduce the quantity $\varphi = (\vartheta - \tau)/\varepsilon$. From the third equation (11.1) we have

$$\varphi = \int_{\psi_0}^{\psi} \frac{\rho_+{}^+ w_+{}^+ \varphi_\psi{}^+}{\rho_+ w_+} \exp\left(-2 \int_{\omega+}^{\omega} \frac{u_+}{w_+} \sin \vartheta d\omega\right) \Big/ d\psi \tag{13.1}$$

From this, using (12.13), we easily see that $w_+$ has order $O(\varepsilon)$, if $\varphi$ has order $O(\varepsilon^2)$. This agrees with the results of [2] (for small angles of attack). There, when $w_+ \geqslant O(\varepsilon)$, i.e. when $\varphi \geqslant O(\varepsilon^2)$, $u_0$ and $w_0$ given by the solution of Gonor are correct; consequently, in this region, the pressure $p_0$ given by formula (12) of [3] is also correct. But then, all the assumptions of [5] are satisfied, and consequently, the pressure $p_0$ given by Gonor holds everywhere. It is easily shown that $\varphi$ defined by (13) in [3] is accurate up to $\varepsilon\varphi$ outside the layer $\varphi = O(\varepsilon^2)$. Thus, the zeroth approximation of Gonor holds everywhere outside the layer $\varphi = O(\varepsilon^2)$.

Then we may determine the function $y_1(\psi)$ from the boundary condition for $u_1$ on the shock surface.

If we now substitute in (13.1) the zeroth approximation for all quantities and substitute $w_0 + \varepsilon w_1$ for $w$ from (12.13), we then obtain the zeroth approximation for $\varphi$, valid everywhere. The formulas thus obtained for $\varphi$ will determine the zeroth approximation of the entropy field in the variables $\varphi$ and $\omega$ everywhere in the region between the shock and cone surfaces.

**14.** We may determine the behavior of the stream lines near the cone surface. From the continuity equation, it follows that for

$$\varphi = O(\varepsilon^2), \quad v_0 = -2u_0(\psi, \omega)\varphi o + O(\varepsilon\varphi) \qquad (u_0(\psi, \omega) = x_0(\psi) + O(\varepsilon))$$

From (12.13), considering Section 13 for $q + \alpha(\psi) = 1/2\pi - O(\varepsilon)$, i.e. in the vortex layer, we have

$$w_+ = -\varepsilon\,[kX_0(\psi)\,p_0(\omega, \tau)]^{-1}\,p_{0\omega}'(\omega, \tau) + o(\varepsilon) \qquad (k = \sin\tau)$$

Then the streamlines near the cone surface in the variables $\varphi$ and $\omega$ are

$$\left(\frac{\varphi}{\varphi_1}\right)\exp\left(2X_0(\psi)\,k\int_{\omega_1}^{\omega} w_1^{-1}\,d\omega\right) = 1 \qquad (14.1)$$

where $\varphi_1$ and $\omega_1$ are the points belonging to the lines $\psi$ in the layer $\varphi = O(\varepsilon^2)$. From (14.1), it follows that all streamlines converge at the point $\varphi = 0$, $\omega = 1/2\pi$, which agrees with the qualitative analysis of A. Ferri.

**15.** There is no difficulty in generalizing to arbitrary conical bodies. Outside the vortex layer, the solution passes to that of Gonor [4]. In the layer $\varphi = O(\varepsilon^2)$, $u_0$ is expressed in a similar manner as for a circular cone, in the formula for $w_1$ the quantity $k$ must be replaced by $A_2$, and in the formula for $v_0$ there is an additional factor $A_1$, where $A_1$ and $A_2$ are the Lamé coefficients computed on the surface of a unit sphere [4]. It is also possible to generalize the above to real gases, if we take as the small parameter the ratio of the densities before and after the shock.

After the present paper was concluded, the author became aware of [7], in which the problem was also solved by the method of PLK; however, the authors of [7] used variables different from those used in this paper, and these variables do not permit them to obtain a single solution up to the cone surface, as was found here.

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## BIBLIOGRAPHY

1. Cheng, H.K., Hypersonic flows past a yawed circular cone and other pointed bodies. *J. Fluid Mech.*, Vol. 12, Part 2, 1962.

2. Sapunkov, Ia.G., Giperzvukovoe obtekanie kruglogo konusa pod uglom ataki (Hypersonic flow past a circular cone at an angle of attack). *PMM* Vol. 27, No. 1, 1963.

3. Gonor, A.L., Obtekanie konusa pod uglom ataki s bol'shoi sverkhzvukovoi skorostiu (High speed supersonic flow past a cone at angle of attack). *Izv. Akad. Nauk SSSR, OTN*, No. 7, 1958.

4. Gonor, A.L., Obtekanie konicheskikh tel pri dvizhenii gaza s bol'shoi sverkhzvukovoi skorostiu (High speed supersonic gas flow past conical bodies). *Izv. Akad. Nauk SSSR, OTN, Mekhanika i mashinostroenie*, No. 1, 1959.

5. Bulakh, B.M., O nesimmetrichnom giperzvukovom obtekanii krugovogo konusa (On asymmetric hypersonic flows past circular cones). *PMM* Vol. 26, No. 5, 1962.

6. Tsian' Siue-sen' (Tsien, H.S.), *Method of Poincaré-Lighthill-Kuo. Problems in Mechanics*, No. 2. IL, 1959. (Also *Advances in Applied Mechanics*, Vol. 4. Academic Press, 1956).

7. Melnik, R.E. and Sheuing, R.A., Shock layer structure and entropy layers in hypersonic conical flows. *Progress Astronaut. and Rocketry*, Vol. 7, pp. 379-420. Academic Press, 1962.

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